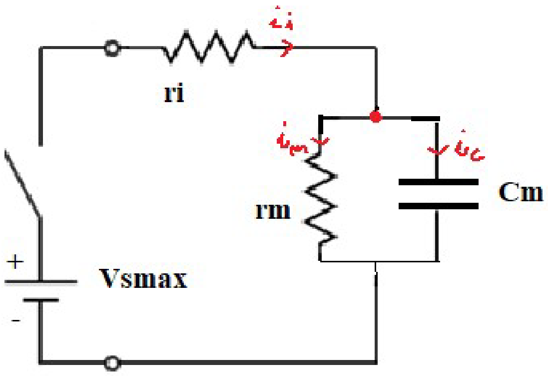
Analysis of the model from a mathematical point of view

Let's consider the first elementary circuit with which we can schematize the axon.
To determine the trend of the potential and currents we use Kirchooff's laws:

$$\mathrm{Vsmax} = V_{r_i} + V_C$$

and

$$i_i = i_c + i_m$$



where Vsmax is the potential across the RC circuit that we imagine as the d.d.p of a generator, Vri is the ddp across the internal resistance of the axon and Vc is the d.d.p ai Condenser C ends. By im we also indicate the current through the membrane.
We derive ic and express everything as a function of the potentials:

$$i_m = \frac{Vc}{r_m}, \qquad i_i = \frac{\mathrm{Vsmax} - \mathbf{Vc}}{r_i}, \qquad i_c = i_i - i_m$$

so what

$$i_c = \frac{\mathbf{Vsmax} - Vc}{r_i} - \frac{Vc}{r_m}$$

Since the following relation holds for a capacitor:

$$Vc = \frac{q}{c_m}$$

we will have

$$i_c = \frac{\mathrm{Vsmax}}{r_i} - \frac{q}{r_i c_m} - \frac{q}{r_m c_m}$$

Deriving the previous expression with respect to time, we obtain:

$$\frac{di_c}{dt} = -\left(\frac{1}{r_i} + \frac{1}{r_m}\right)\frac{i_c}{c_m}$$

which is a linear differential equation with separable variables whose solution is obtained by integrating:

$$\int \frac{di_c}{i_c} = -\left(\frac{1}{r_i} + \frac{1}{r_m}\right)\frac{1}{c_m}\int dt$$

$$i_c(t) = Ae^{-\frac{t}{\tau}}$$

The time constant τ is

$$\tau = \frac{r_i r_m c_m}{r_i + r_m}$$

The constant A is determined by the initial conditions of the circuit.

For

$$\textbf{t=0} \quad i_c(0) = A = i_i(0) = \frac{Vsmax}{r_i}$$

and then

$$i_c(t) = \frac{Vsmax}{r_i} e^{-\frac{t}{\tau}}$$

Usually rm ~ 104 Ω/m and ri ~ 1010 Ωμ and rm can be neglected compared to ri, which allows to simplify τ:

$$\tau = \frac{r_i r_m c_m}{r_i + r_m} \sim r_m c_m$$

We then find the other currents as a function of time:

$$i_m(t) = \frac{Vc}{r_m} = \int \frac{i_c(t)}{r_m c_m} dt$$

The solution is:

$$i_m(t) = \frac{Vsmax}{r_i + r_m}\left(1 - e^{-\frac{t}{\tau}}\right)$$

In the end

$$i_i(t) = i_c(t) + i_m(t) = \frac{Vsmax}{r_i + r_m}\left(1 + \frac{r_m}{r_i} e^{-\frac{t}{\tau}}\right)$$

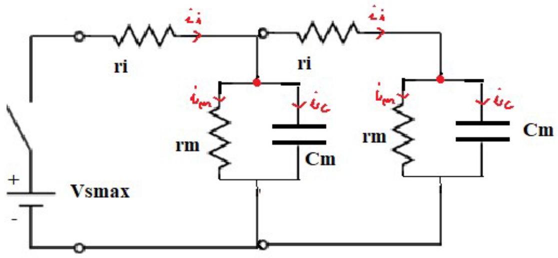
For t=0 all the current passes through the resistor ri and goes to charge the capacitor cm. No current circulates through rm, im(0) = 0, in accordance with the fact that it is in parallel with cm and has no d.d.p. at its ends.
For long times, (t→∞) the capacitor is charged and no longer circulates current through the branch i.e. ic(∞) = 0, while all the current passes through the internal and membrane resistance:

$$i_i(\infty) = i_m(\infty) = \frac{Vsmax}{r_i+r_m}$$

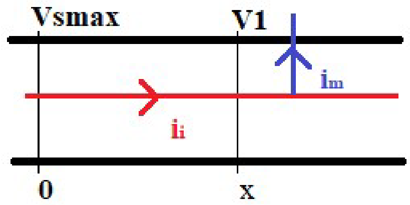
The expressions found analytically for currents justify a result that we had previously introduced, namely the stimulus potential:

$$V_s(t) = V_{smax}(1 - e^{-\frac{t}{\tau}})$$

To simulate the axon, we consider other similar circuits connected to the first. For this second circuit, the potential or f.e.m. is no longer Vsmax but the potential present to the heads of the parallel



$$V_c = i_m r_m = \frac{Vsmax r_m}{r_i + r_m} \sim \frac{r_m}{r_i} Vsmax$$



The stimulus potential decreases along the axon.
Let us consider the stimulus potential Vsmax at the point x=0. Its propagation depends on the current flowing within axon II, which in turn depends (according to Ohm's first law) on the variation of potential along the axon:

$$i_i(x) = -\frac{1}{r_i}\frac{dVs}{dx}$$

The minus sign depends on the fact that the current flows from points to higher potentials points with lower potentials, then in the opposite direction to the gradient of Vs. The internal current decreases because the charges flow out of the membrane via the current im:

$$\frac{di_i}{dx} = -i_m = -\frac{Vs}{r_m}$$

Substituting we get the following second-order differential equation:

$$Vs = -r_m \frac{di_i}{dx} = \frac{r_m}{r_i}\frac{d^2Vs}{dx^2}$$

The solution is:

$$Vs(x) = Vsmaxe^{-\frac{x}{\lambda}}$$

with

$$\lambda = \sqrt{\frac{r_m}{r_i}}$$

as we had previously introduced.